

QFT in the flat chart of de Sitter space

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Abstract

We study the correlators for interacting quantum field theory in the flat chart of de Sitter space at all orders in perturbation. The correlators are calculated in the in-in formalism which are often applied to the calculations in the cosmological perturbation. It is shown that these correlators are de Sitter invariant. They are compared with the correlators calculated based on the Euclidean field theory. We then find that these two correlators are identical. This correspondence has been already shown graph by graph but we give an alternative proof of it by direct calculation.

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1 Introduction

In recent years, there have been rapid progresses in the precise measurement of the observable quantities in cosmology, e.g., the non-Gaussianity of the fluctuations generated during inflation, which is expected to be a powerful tool as a probe of the early universe. Along with the development of these precise measurements, the need arises for the accurate theoretical predictions of the corresponding quantities.

When computing the non-Gaussianity, one needs to discuss interacting quantum field theory on inflationary background, in which one does not generally know how to define the interacting vacuum. One often uses the $i\epsilon$ prescription in cosmology to calculate the correlators perturbatively. (See, for example, Ref. [1].) In the Minkowski, this prescription is known to perturbatively give the Poincaré invariant correlators for interacting theory, defining interacting vacuum as the lowest energy eigen state. Indeed, this prescription also enables us to calculate the non-Gaussianity or higher correlations in the inflationary era, but the physical meaning of it is not as clear as in the Minkowski case. Our main interest in this paper is in the meaning of the $i\epsilon$ prescription for interacting field theory in de Sitter space.

The free scalar quantum field theory in de Sitter space is well understood [2–5], while interacting one is a hot subject with a lot of debate [6–43]. We focus in the present paper on the problem whether the $i\epsilon$ prescription for interacting theory breaks de Sitter invariance.

Since de Sitter space is maximally symmetric and possesses $SO(4, 1)$, de Sitter symmetry, it is strongly expected to exist a de Sitter invariant vacuum even for interacting theory. In fact, a de Sitter invariant vacuum for interacting theory is defined by constructing arbitrary correlators perturbatively at all orders using the Euclidean method [10]. While the vacuum state thus constructed are manifestly de Sitter invariant, it is not obvious whether the ones defined by the $i\epsilon$ prescription in the flat chart are de Sitter invariant. Notice, for example, that in the latter the integration region for the vertices in calculating correlators are restricted to the future of the cosmological horizon, which is not de Sitter invariant.

Actually, this problem has been already resolved affirmatively in Ref. [11] for interacting massive scalar field. Namely, the $i\epsilon$ prescription does not break de Sitter invariance for interacting massive scalar field. Furthermore, the vacuum defined by the $i\epsilon$ prescription has been shown to be equivalent to the

Euclidean vacuum. The main ideas in Ref. [11] are as follows. They start from correlators defined on an Euclidean sphere and take, on the Euclidean sphere, coordinates such that when we Wick rotate the time coordinate continues to the static chart of the Lorentzian de Sitter space. Then, after the deformation of the integral path of the Euclidean time, fall-off of the propagator in the large separation limit leads to the identity of the two correlators at least on the static chart. From the analyticity of the in-in correlators for their time coordinates, and the uniqueness of the analytic continuation, it is shown that the in-in correlators in the flat chart are identical to the analytic continuations of those on an Euclidean sphere.

Then, it is natural to ask what becomes of in massless field theory. What happens for graviton in de Sitter space has especially been a topic of much discussion. (See, e.g. [13, 21–23].) Our final goal is to extend the correspondence between the two vacua to those interacting massless field theory. It is also worth considering derivatively interacting massless scalar field, which can be a step toward graviton.

It seems difficult to extend the discussion of massive field theory above to massless field theory where the propagator does not fall off in general, since the proof of the correspondence between the two vacua relies on this decay property of the propagator at a large separation as explained above. In order to attack those theories, we take another approach. That is, we directly calculate the correlators with the $i\epsilon$ prescription. We derive, along this way, the analytic Mellin-Barnes formulae for the correlators of quantum fields in the flat chart. The resulting correlators are shown to be completely the same as the analytic continuations of the ones considered in the Euclidean field theory in Ref. [10]. Thus we find that the $i\epsilon$ prescription in de Sitter space gives the vacuum state corresponding to the Euclidean field theory. Although we consider only massive theory in the present paper, we believe that our proof has potential to be extended to wider range of theories which include interacting massless theory such as derivatively interacting one, since it does not employ the decay property of the propagator.

This paper is organized as follows. In Sec. 2, we briefly review how to describe de Sitter space, especially the flat chart, and massive free scalar quantum field theory on it. Pauli-Villars regularization scheme is also introduced. Then we proceed to the interacting theory, in Sec. 3, 4 and 5. We consider, in Sec. 3 and 4, a tree graph which contributes to an N -pt correlator with single vertex. Then in Sec. 5, we extend the discussion to arbitrary graphs. We give a brief summary in Sec. 6.

2 Preliminaries

In this section, we briefly review free scalar quantum field theory on de Sitter space, especially in the flat chart. We also introduce Pauli-Villars regularization scheme for later use.

2.1 de Sitter space

We consider D -dimensional de Sitter space dS^D with, for simplicity, unit radius. This is a hyperboloid embedded in $(D + 1)$ -dimensional Minkowski space with metric $\eta_{ab} = (-, +, \dots, +)$. The embedding is specified by

$$\eta_{ab}X^aX^b = 1. \quad (2.1)$$

It is convenient to define the invariant distance between two points X and Y in de Sitter space by the Minkowski inner product of X and Y , which we denote as

$$Z(X, Y) := \eta_{ab}X^aY^b, \quad (2.2)$$

as in Ref. [9]. For brevity, we often use alternative notation Z_{XY} for $Z(X, Y)$, Z_{1Y} for $Z(X_1, Y)$ and so forth in the following.

The coordinates (η, \mathbf{x}) in the flat chart are related to the embedding coordinates as

$$X^0 = \frac{1}{2} \left(\eta - \frac{1}{\eta} \right) - \frac{\|\mathbf{x}\|^2}{2\eta}, \quad X^D = -\frac{1}{2} \left(\eta + \frac{1}{\eta} \right) + \frac{\|\mathbf{x}\|^2}{2\eta}, \quad X^\alpha = -\frac{x^\alpha}{\eta}, \quad (\alpha = 1, 2, \dots, D-1), \quad (2.3)$$

where $\|\mathbf{x}\|$ means the norm of $(D-1)$ -vector \mathbf{x} . The flat chart coordinates with $-\infty < \eta < 0$ and $\mathbf{x} \in \mathbf{R}^{D-1}$ span just a half of the whole spacetime region. In fact, the linear combination

$$X^0 + X^D = -\frac{1}{\eta}, \quad (2.4)$$

is restricted to the positive side for negative η . The metric in the flat chart is expressed as

$$ds^2 = \frac{1}{\eta^2} (-d\eta^2 + d\mathbf{x}^2). \quad (2.5)$$

Expressed in the flat chart coordinates, the invariant distance between X and X' , $Z(X, X')$ is given by

$$Z(X, X') = 1 + \frac{(\eta - \eta')^2 - \|\mathbf{x} - \mathbf{x}'\|^2}{2\eta\eta'}, \quad (2.6)$$

where (η, \mathbf{x}) and (η', \mathbf{x}') are the flat chart coordinates corresponding to X and X' , respectively.

2.2 Free QFT on de Sitter

We now consider a massive free scalar QFT on de Sitter space. We focus on the Green's function $G(X, Y)$ given by

$$G(X, Y) = \frac{\Gamma(-\sigma)\Gamma(\sigma + D - 1)}{(4\pi)^{D/2}\Gamma(D/2)} {}_2F_1 \left(-\sigma, \sigma + D - 1; \frac{D}{2}; \frac{1 + Z_{XY}}{2} \right), \quad (2.7)$$

which corresponds to taking Bunch-Davies vacuum [44] or Euclidean vacuum [45]. σ is related to the mass of the field m by

$$\sigma = -\frac{D-1}{2} + \sqrt{\left(\frac{D-1}{2}\right)^2 - m^2}. \quad (2.8)$$

Expressing the hypergeometric function in the Barnes representation, we have

$$G(X, Y) = \int_{\nu} \left(\frac{1 - Z_{XY}}{2} \right)^{\nu} \Gamma(-\nu) \psi(\nu), \quad (2.9)$$

with

$$\psi(\nu) := \frac{1}{(4\pi)^{D/2}} \Gamma \left[\begin{matrix} -\sigma + \nu, \sigma + D - 1 + \nu, 1 - \frac{D}{2} - \nu \\ \frac{D}{2} + \sigma, 1 - \frac{D}{2} - \sigma \end{matrix} \right]. \quad (2.10)$$

Here

$$\Gamma \left[\begin{matrix} \alpha_1, \alpha_2, \dots \\ \beta_1, \beta_2, \dots \end{matrix} \right],$$

stands for $\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots/\Gamma(\beta_1)\Gamma(\beta_2)\cdots$, and the symbol $\int_{\nu}(\cdots)$ means the Barnes integral. The Barnes integral is an integral along a straight line, C , that traverses from $-i\infty$ to $+i\infty$ parallel to the

imaginary axis with the factor $1/2\pi i$:

$$\int_{\nu}(\cdots) := \int_C \frac{d\nu}{2\pi i}(\cdots) . \quad (2.11)$$

The integrand of the Barnes integral includes sequences of poles. For example, $\Gamma(z)$ possesses a sequence of poles at $z = 0, -1, -2, \cdots$. The integration path C is taken to avoid all the sequences of poles in the integrand. In the case of the above Green's function, C is taken to satisfy

$$\max\{-\operatorname{Re} \sigma - D + 1, \operatorname{Re} \sigma\} < \operatorname{Re} \nu < \min\left\{1 - \frac{D}{2}, 0\right\} . \quad (2.12)$$

This region of the integration path is called “fundamental strip,” and the poles such that are associated with Gamma functions like $\Gamma(\cdots - \nu)$ ($\Gamma(\cdots + \nu)$) and hence such that line up on the right (left) hand side of this strip are called right (left) poles. (See Fig. 1.) The symbol like \int_{ν} is used to represent the Barnes integral in this meaning in the following.

2.3 Pauli-Villars Regularization

Because we consider interacting theory in the present paper, we have to introduce some ultra-violet regularization scheme. We make use of the Pauli-Villars regularization. This scheme attaches some massive propagators, $G_i(X, Y)$, defined in Eq. (2.9) with m replaced by the regulator mass M_i , to the original one, $G(X, Y)$, so that we replace the original propagator in a graph with the regularized propagator

$$G^{\text{reg}}(X, Y) := G(X, Y) + \sum_i C_i G_i(X, Y) . \quad (2.13)$$

The coefficients C_i are chosen so that the regularized propagator $G^{\text{reg}}(X, Y)$ becomes finite in the coincidence limit $Y \rightarrow X$, which leads to the conditions

$$\sum_i C_i = -1 , \quad \sum_i C_i M_i^2 = 0 , \quad \sum_i C_i M_i^3 = 0 , \quad \cdots . \quad (2.14)$$

This regularization scheme affects the pole structure of $\psi(\nu)$ in (2.9), eliminating the first several right poles of $\psi(\nu)$ which are responsible for the behaviour of the Green's function in the coincidence limit [10]. The regularized Green's function is written as

$$G^{\text{reg}}(X, Y) = \int_{\nu} \left(\frac{1 - Z_{XY}}{2} \right)^{\nu} \Gamma(-\nu) \psi^{\text{reg}}(\nu) , \quad (2.15)$$

where we assume that $\psi^{\text{reg}}(\nu)$ is regularized to be analytic in the region

$$\operatorname{Re} \sigma < \operatorname{Re} \nu < p , \quad (2.16)$$

with p a sufficiently large positive constant. (See Fig. 1.) In the following sections, we drop, for simplicity, the symbols such as reg on G and ψ .

3 Interacting QFT: Single Vertex

We now move on to the interacting theory. The interacting QFT in the flat chart of the Lorentzian de Sitter space is discussed in the present and the succeeding sections. When we express the correlators in the wave number representation, we employ the $i\epsilon$ prescription to calculate the correlators for the interacting vacuum. This prescription regularizes the oscillatory behaviour of the Green's functions at

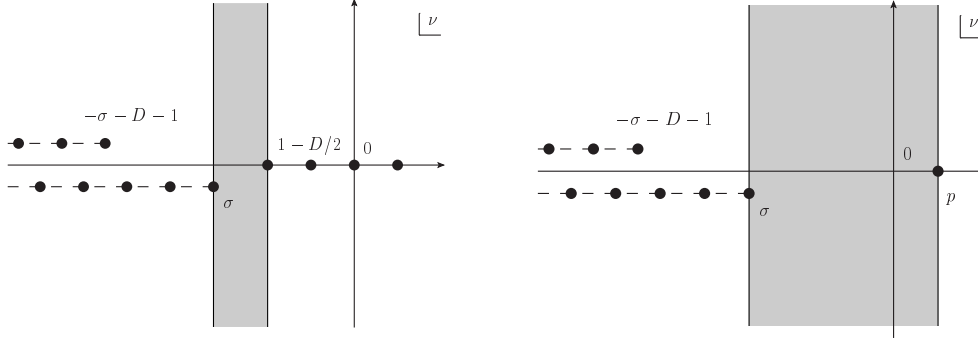


Figure 1: The left figure shows the pole structure for $\psi(\nu)$ which is not regularized. There are two series of left poles from $\nu = \sigma$ and $\nu = -\sigma - D - 1$, and right poles from $\nu = 1 - D/2$. The right one shows the pole structure for $\psi^{\text{reg}}(\nu)$ which is Pauli-Villars regularized. The shaded region represents the fundamental strip in each figure.

infinity in time and makes the vertex integral converge. Although what we discuss in the present paper is the position space representation of the correlators, we also employ the $i\epsilon$ prescription to specify the interacting vacuum.

In this section, we discuss perturbative calculations of a single vertex tree graph for the correlators. Then, we identify the problems to be solved to accomplish this calculation, which are solved in Sec. 4. In Sec. 5 the results for single vertex tree graphs are extended to arbitrary graphs.

3.1 Definition of the In-in Path

Let us consider N -pt Green's function. The contribution to N -pt correlator at the lowest order in perturbation theory is given by

$$\mathcal{V}_N(X_1, \dots, X_N) = \int_{\Omega} dV_Y G(X_1, Y) \cdots G(X_N, Y). \quad (3.1)$$

In the in-in formalism with the $i\epsilon$ prescription, the integration region Ω for the vertex integral is specified as follows.

We first introduce an η -integration path P on the η -plane, independently of the spatial coordinates \mathbf{y} , defined as a curve which starts from $-\infty e^{-i\epsilon}$ and ends at $-\infty e^{i\epsilon}$ as shown in Fig. 2. All the external points are also supposed to be placed along this path. In case of the wave number representation, this construction completes the definition of the in-in path on the η -plane. If we take the η -path along P , the integral converges with the integrand vanishing fast enough in the past.

For the purpose of the present paper, it is more convenient to use the position space representation to compute the correlators. The vertex integrals involve the spatial integration, too. As a starting point, we set the region of the vertex integral Ω to $P \times \mathbf{R}^{D-1}$. If we first carry out the spatial integration before temporal one, the integral would diverge because we then pick up the contributions from distant spacelike region. On the other hand, if we integrate first for the time variable and then for the spatial ones, the integral is convergent as we see in Sec. 4. This means that the integral over $P \times \mathbf{R}^{D-1}$ is not well-defined as a multiple integral.

To make the integral to be well-defined as a multiple integral, we modify the integral region by deforming the path of the η -integral P [11]. There are branching points on the η -plane, which correspond to the intersections with the light cones emanating from the external points. On the η -plane for fixed \mathbf{y} , $G(X_i, Y)$ has the same structure of Riemann surface as that of $(1 - Z_{iY})^{\nu_i}$, where ν_i is some complex

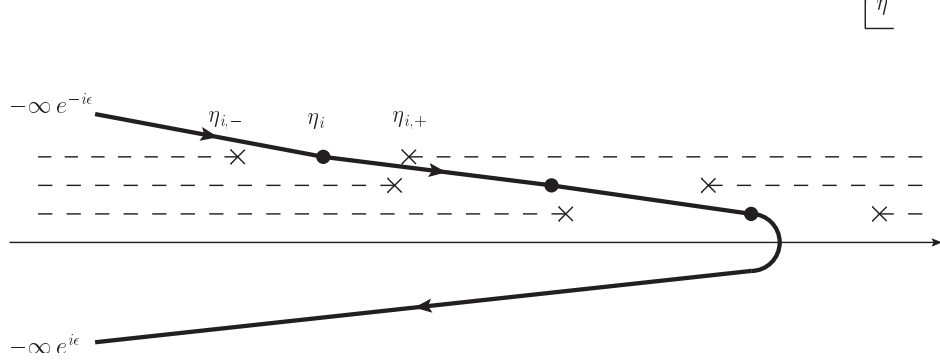


Figure 2: This figure shows the η -path P which is later deformed to $P_{\mathbf{y}}$. The dots represent the time coordinate η_i of the external points and the crosses are the branching points corresponding to the light cones emanating from the external points. The dashed lines are the branch cuts.

number, and

$$\frac{1 - Z_{iY}}{2} = \frac{(-\eta + \eta_{i,+})(\eta - \eta_{i,-})}{4(-\eta)(-\eta_i)} , \quad (3.2)$$

where

$$\eta_{i,\pm} := \eta_i \pm \|\mathbf{x}_i - \mathbf{y}\| , \quad (i = 1, \dots, N-1, N) . \quad (3.3)$$

Namely, the integrand has the same structure of Riemann surface as that of

$$(-\eta)^{-(D+\sum \nu_i)} \prod_{i=1}^N (-\eta + \eta_{i,+})^{\nu_i} (\eta - \eta_{i,-})^{\nu_i} . \quad (3.4)$$

The time integration is unchanged even if we deform the integration contour as long as it does not cross singularities of the integrand. Thus, we deform the contour P to $P_{\mathbf{y}}$ such that the maximum value of the real part of η on $P_{\mathbf{y}}$ is equal to $\max_i \{\text{Re } \eta_{i,-}\} + b$ where b is a small real positive constant. (See Fig. 3.) This deformation on the η -plane is significant when the spatial coordinates of the vertex is largely separated from those of relevant external points. To the contrary, when $\|\mathbf{x}_i - \mathbf{y}\|$ is small for i that realizes the maximum among $\text{Re } \eta_{i,-}$, the modified contour $P_{\mathbf{y}}$ is almost identical to the original one P .

Using this $P_{\mathbf{y}}$, we define the integration region

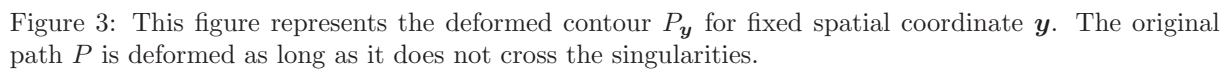
$$\Omega := \{(\eta, \mathbf{y}) \mid \eta \in P_{\mathbf{y}}, \mathbf{y} \in \mathbf{R}^{D-1}\} , \quad (3.5)$$

in $\mathbf{C} \times \mathbf{R}^{D-1}$. The result of the integral is the same as that is obtained by integrating first for time and then for space for the original integration region, but we emphasize that the integral over Ω is now a multiple integral.

3.2 Problems to Be Solved in the Calculations

Let us return to Eq. (3.1). Inserting Eq. (2.9) into Eq. (3.1), we have

$$\mathcal{V}_N(X_1, \dots, X_N) = \int_{\Omega} dV_Y \int_{\nu_1} \dots \int_{\nu_N} \left[\prod_{i=1}^N \Gamma(-\nu_i) \psi(\nu_i) \right] \times \left[\prod_{i=1}^N \left(\frac{1 - Z_{iY}}{2} \right)^{\nu_i} \right] . \quad (3.6)$$



If we can exchange the order of the integrals, $\int_{\Omega} dV_Y$ and $\int_{\nu_1} \cdots \int_{\nu_N}$, we are led to calculate the following integral

The first problem is to calculate this integral. This quantity is shown to have an analytic Mellin-Barnes representation in Sec. 4, and hence if this exchange of the order of integration is allowed, \mathcal{V}_N can be represented in an analytic Mellin-Barnes form. It is not trivial whether this exchange of the order of the integration is allowed or not. This is the second problems. The same problem arises also for arbitrary graphs as for the tree level graphs. We will extend our discussion to arbitrary graphs in Sec. 5.

The goal of this section is to compute the master integral:

where we have introduced $n := N - 1$ for convenience,

is the invariant volume, and Ω is defined in Eq. (3.5).

In order to evaluate the above expression (4.1), we introduce the following generating function

following Ref. [10], in which it was used to evaluate the master integral on an Euclidean sphere. Here

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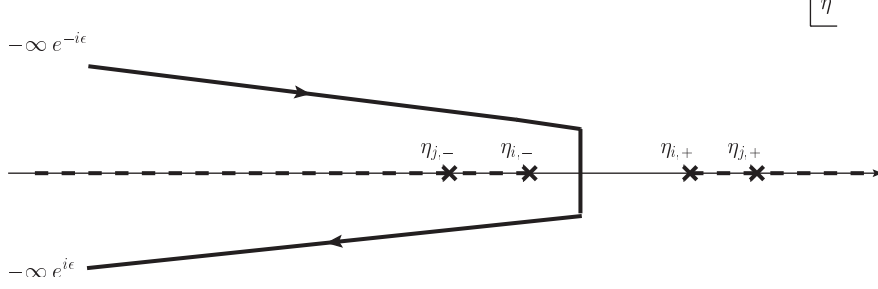


Figure 4: A figure representing the in-in path $P_{\mathbf{y}}$ for the external points which lie on the real Lorentzian section and are mutually spacelike separated. The crosses represent the branching points and the dotted lines the branch cuts.

are assumed.

In this subsection we establish the relation between the generating function and the master integral. Formally, in the same way as in the Euclidean case discussed in Ref. [10], the generating function (4.3) seems to be related to the master integral (3.7) also in the present case as follows:

[Step 1.] We first apply Eq. (A.1) to the integrand of (4.3) to obtain

$$\begin{aligned} \mathcal{A}(\alpha_1, \dots, \alpha_n) &= \int_{\Omega} dV_Y \frac{1}{\Gamma(-\lambda)} \int_{u_1} (\alpha_1)^{u_1} \dots \int_{u_n} (\alpha_n)^{u_n} \\ &\quad \times \Gamma[-u_1, \dots, -u_n, -u_N] \left(\frac{1 - Z_{1X}}{2} \right)^{u_1} \dots \left(\frac{1 - Z_{NX}}{2} \right)^{u_N}, \end{aligned} \quad (4.5)$$

where

$$u_N := \lambda - \sum_{i=1}^n u_i. \quad (4.6)$$

[Step 2.] Next, we exchange the order of the integration, $\int_{\Omega} dV_Y$ and $\int_{u_1} \dots \int_{u_n}$, to have

$$\begin{aligned} \mathcal{A}(\alpha_1, \dots, \alpha_n) &= \frac{1}{\Gamma(-\lambda)} \int_{u_1} (\alpha_1)^{u_1} \dots \int_{u_n} (\alpha_n)^{u_n} \\ &\quad \times \Gamma[-u_1, \dots, -u_n, -u_N] \mathcal{M}(u_1, \dots, u_n, u_N). \end{aligned} \quad (4.7)$$

Thus, the Mellin transform of \mathcal{A} gives \mathcal{M} .

However, we have to prove that [Step 1.] and [Step 2.] are indeed possible, which is the goal of this subsection. In particular, [Step 2.] requires that the integral over Ω is a multiple integral. The convergence of the integral is rather obvious when we consider the corresponding integral over a compact Euclidean sphere, while it is not in the present case where the integration region is non-compact. In this subsection, we assume, for a technical reason, that the time coordinates of all external points lie on the real Lorentzian section, i.e. $\eta_i \in \mathbf{R}_-$, $\mathbf{y}_i \in \mathbf{R}^{D-1}$, and furthermore, that any pairs of them are mutually spacelike separated.

Since the definition of the in-in path described in Sec. 3.1 requires the external points to lie along the in-in path and therefore their time coordinates are complex in general, we need some explanations of the in-in path for this configuration. The path is defined on the η -plane by taking the limit $\text{Im } \eta_i \rightarrow 0$ in $P_{\mathbf{y}}$ introduced in Sec. 3.1. It seems that the path in this limit must, at least partly, lie on the η -real axis. However, since the external points are mutually spacelike, the branch cuts, lying on the η -real axis, do not cover the whole η -real axis. Therefore, the limit can be taken without the path $P_{\mathbf{y}}$ crossing the branch cuts, and hence the in-in path in this limit is simply a contour going from $-\infty e^{-i\epsilon}$ to $-\infty e^{i\epsilon}$ as

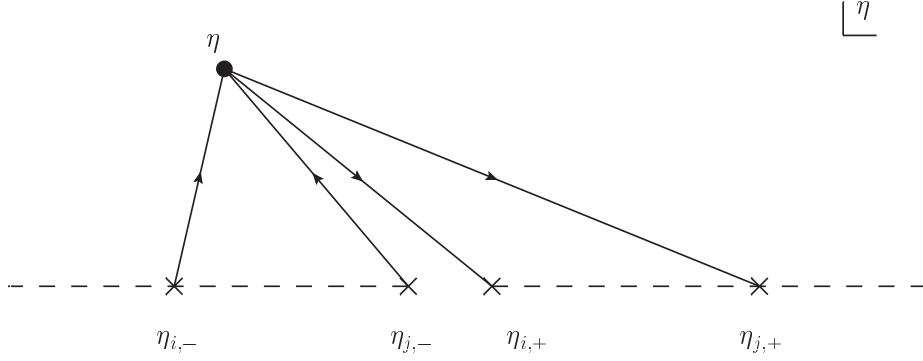


Figure 5: The dot represents the time coordinate of a vertex on $P_{\mathbf{y}}$. The remainder of the summation of the arguments of two vectors relevant to the subscript i and that relevant to the subscript j gives $|\arg(1 - Z_{iY}) - \arg(1 - Z_{jY})|$ as in (4.10).

shown in Fig. 4.

Proof of [Step 1.]: Note that the following inequalities hold for arbitrary $Y \in \Omega$:

$$|\arg(1 - Z_{iY}) - \arg(1 - Z_{jY})| < \pi, \quad (i, j = 1, 2, \dots, N). \quad (4.8)$$

In fact, $\arg(1 - Z_{iY})$ is given by

$$\arg(1 - Z_{iY}) = \arg(-\eta + \eta_{i,+}) + \arg(\eta - \eta_{i,-}) - \arg(-\eta) - \arg(-\eta_i), \quad (4.9)$$

and then, noticing that $\arg(-\eta_i) = 0$ since all the external points are on the real Lorentzian section, we have

$$\begin{aligned} & |\arg(1 - Z_{iY}) - \arg(1 - Z_{jY})| \\ &= |\arg(-\eta + \eta_{i,+}) + \arg(\eta - \eta_{i,-}) - \arg(-\eta + \eta_{j,+}) - \arg(\eta - \eta_{j,-})|. \end{aligned} \quad (4.10)$$

This quantity is less than π for any $(\eta, \mathbf{y}) \in \Omega$. (See Fig. 5.) The inequality (4.8) is the sufficient condition that the formula (A.1) can be applied to the integrand of Eq. (4.3). For the later purpose, we modify the integration path $P_{\mathbf{y}}$ as such that satisfies

$$|\arg(1 - Z_{iY}) - \arg(1 - Z_{jY})| < \pi - \delta, \quad (4.11)$$

for any i and j with a small positive number δ . This can be achieved easily. Because $|\arg(1 - Z_{iY}) - \arg(1 - Z_{jY})|$ is close to π only in the small region surrounding the interval $(\eta_{i,-}, \eta_{j,-})$ or $(\eta_{i,+}, \eta_{j,+})$, the path can be chosen to avoid this region.

Proof of [Step 2.]: We denote the integration paths for u_1, \dots, u_n as C_1, \dots, C_n , respectively, and define $C := C_1 \times \dots \times C_n$. The sufficient condition to allow to exchange the order of the integration, $\int_{\Omega} dV_Y$ and $\int_C \prod_{i=1}^n du_i / 2\pi i$, is that the integral is absolutely convergent (Fubini's theorem). In the present case, we should examine the following integral

$$\begin{aligned} & \frac{1}{|\Gamma(-\lambda)|} \int_C \prod_{k=1}^n \left| \frac{du_k}{2\pi i} \right| |(\alpha_1)^{u_1} \dots (\alpha_n)^{u_n}| |\Gamma[-u_1, \dots, -u_n, -u_N]| \\ & \times \left[\int_{\Omega} |dV_Y| \left| \left(\frac{1 - Z_{1Y}}{2} \right)^{u_1} \right| \dots \left| \left(\frac{1 - Z_{NY}}{2} \right)^{u_N} \right| \right], \end{aligned} \quad (4.12)$$

where

$$|dV_Y| = \frac{|d\eta|d^{D-1}y}{|-\eta|^D} . \quad (4.13)$$

If this integral is finite, then we can justify the exchange of the order of integrals in [Step 2].

To show this, we focus on the integrand of the Ω integral in the large brackets in Eq. (4.12) for fixed u_1, \dots, u_n :

$$\left| \left(\frac{1 - Z_{1Y}}{2} \right)^{u_1} \right| \dots \left| \left(\frac{1 - Z_{NY}}{2} \right)^{u_N} \right| . \quad (4.14)$$

Notice that

$$|(1 - Z_{iY})^{u_i}| = |1 - Z_{iY}|^{\text{Re } u_i} \exp[-\arg(1 - Z_{iY})\text{Im } u_i] . \quad (4.15)$$

Along the integration path of u_i parallel to the imaginary axis, $\text{Im } u_i$ varies while $\text{Re } u_i$ is fixed. Taking into account that u_N includes u_i as given in Eq. (4.6), the part depending on $\text{Im } u_i$ in Eq. (4.14) is factored out as

$$\exp[\{\arg(1 - Z_{NY}) - \arg(1 - Z_{iY})\}\text{Im } u_i] . \quad (4.16)$$

Since $|\arg(1 - Z_{NY}) - \arg(1 - Z_{iY})|$ is bounded as shown in Eq. (4.11), this factor is bounded from above by $\exp[(\pi - \delta)|\text{Im } u_i|]$. Therefore, noticing that α_i is real positive number, we find that

$$\begin{aligned} [\text{Eq. (4.12)}] &< \frac{1}{|\Gamma(-\lambda)|} |(\alpha_1)^{\text{Re } u_1} \dots (\alpha_n)^{\text{Re } u_n}| \int_{\Omega} |dV_Y| \left| \left(\frac{1 - Z_{1Y}}{2} \right)^{\text{Re } u_1} \right| \dots \left| \left(\frac{1 - Z_{NY}}{2} \right)^{\text{Re } u_N} \right| \\ &\quad \times \int_C \prod_{k=1}^n \left| \frac{du_k}{2\pi i} \right| |\Gamma[-u_1, \dots, -u_n, -u_N]| e^{(\pi - \delta)(|\text{Im } u_1| + \dots + |\text{Im } u_n|)} . \end{aligned} \quad (4.17)$$

Since $|\Gamma(x + iy)| \approx (2\pi)^{1/2} e^{-\pi|y|/2} |y|^{x-1/2} (|y| \rightarrow +\infty)$, u_k integrals in the second line in the last expression are convergent. Therefore, our remaining task is to show that the volume integral

$$\int_{\Omega} |dV_Y| \left| \left(\frac{1 - Z_{1Y}}{2} \right)^{\text{Re } u_1} \right| \dots \left| \left(\frac{1 - Z_{NY}}{2} \right)^{\text{Re } u_N} \right| , \quad (4.18)$$

is also finite.

For this purpose, we first introduce a representative point X_0 with coordinates in the flat chart defined by

$$(\eta_0, \mathbf{x}_0) := \sum_{i=1}^N p_i (\eta_i, \mathbf{x}_i) , \quad (p_i \geq 0, \sum p_i = 1) , \quad (4.19)$$

and a domain D_0 far from X_0 in terms of the invariant distance by

$$D_0 := \{Y \mid |Z_{0Y}| > Z_0\} \cap \Omega . \quad (4.20)$$

Note that if we take Z_0 to be sufficiently large, we see that

$$|Z_{iY}| \geq \text{const.} \times |Z_{0Y}| , \quad (Y \in D_0, i = 1, 2, \dots, N) . \quad (4.21)$$

We divide the region Ω into $(\Omega \setminus D_0)$ and D_0 , and evaluate each contribution to (4.18) separately.

(i) Integral over $\Omega \setminus D_0$:

We further divide $\Omega \setminus D_0$ into K defined by

$$K := \{(\eta, \mathbf{y}) \in \Omega \setminus D_0 \mid \|\mathbf{x}_0 - \mathbf{y}\| > R\} , \quad (4.22)$$

and its complement $(\Omega \setminus D_0) \setminus K$. R is set large enough for K not to include any external points. (See Fig. 6.)

(i-a) Integral over $(\Omega \setminus D_0) \setminus K$:

The region $(\Omega \setminus D_0) \setminus K$ is compact but it contains the coincidence points $(\eta, \mathbf{y}) = (\eta_i, \mathbf{x}_i)$ at which the integrand of (4.18) diverges. Since $\mathbf{y} \approx \mathbf{x}_i$ around them, the path $P_{\mathbf{y}}$ is identical to the original one P , and hence $\eta = \eta_i + i\delta\eta$ with real $\delta\eta$. Then, we have

$$|1 - Z_{iY}|^{\text{Re } u_i} \approx ((\delta\eta)^2 + \|\mathbf{x}_i - \mathbf{y}\|^2)^{\text{Re } u_i} \quad (4.23)$$

around a point (η_i, \mathbf{x}_i) , which shows that (4.18) is finite as long as we choose the integration path of u_i to satisfy

$$\text{Re } u_i > -D/2 , \quad (i = 1, 2, \dots, n, N) , \quad (4.24)$$

which does not conflict with [Step 1.]. Recall that the fundamental strip of Eq. (4.5) contains the paths with $\text{Re } u_i$ for all i being infinitesimally small negative constants.

(i-b) Integral over K :

We first see that, for $Y \in K$, $|1 - Z_{iY}|$ is bounded both from below and from above by positive constants. Recall that the η -path $P_{\mathbf{y}}$ is defined by deforming P not to touch $\eta_{i,-}$ except for the case with $\mathbf{y} \approx \mathbf{x}_i$, which occurs in $(\Omega \setminus D_0) \setminus K$. Therefore, $|1 - Z_{iY}|$ does not vanish, bounded from below by some constant $c_- (> 0)$. It is also easy to show that $|1 - Z_{iY}|$ is bounded from above by some constant c_+ . If $|1 - Z_{iY}|$ is sufficiently large, Z_{0Y} will be larger than Z_0 . Then, by the definition of K , Y is not included in K . Thus, we conclude that for some positive constants c_{\pm} ,

$$c_- < |1 - Z_{iY}| < c_+ , \quad (Y \in K) . \quad (4.25)$$

Furthermore, one can claim that the volume of the region K is finite, i.e.,

$$\int_K |dV_Y| < +\infty . \quad (4.26)$$

In showing this, the non-trivial point is that the region K extends to infinitely large $\|\mathbf{y}\|$. However, the region of the η -integral is confined to the interval

$$\eta_{i(\mathbf{y}),-} - b' \leq \text{Re } \eta \leq \eta_{i(\mathbf{y}),-} + b , \quad (4.27)$$

where b is the same constant used in defining the path $P_{\mathbf{y}}$ and $i(\mathbf{y})$ is the label of the external point such that $\eta_{i(\mathbf{y}),-} > \eta_{k,-}$ for all $k \neq i(\mathbf{y})$. Here the point is that one can choose a large positive constant b' to be independent of \mathbf{y} . In fact, the invariant distance between $X_0 := (\eta_0, \mathbf{x}_0)$, and the point corresponding to the above lower bound $Y_{\text{bdry}} := (\eta_{i(\mathbf{y}),-} - b', \mathbf{y}) = (\eta_{i(\mathbf{y})} - \|\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}\| - b', \mathbf{y})$ is evaluated as

$$\begin{aligned} |1 - Z(X_0, Y_{\text{bdry}})| &= \left| \frac{(\eta_0 - \eta_{i(\mathbf{y})} + \|\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}\| + b')^2 - \|\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}\|^2}{2\eta_0(\eta_{i(\mathbf{y})} - \|\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}\| - b')} \right| \\ &\gtrsim \frac{b'}{|\eta_0|} , \quad (\|\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}\| + b' \rightarrow +\infty) . \end{aligned} \quad (4.28)$$

In the last inequality we assumed $\|\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}\| + b' \rightarrow +\infty$ but this should be a good approximation in

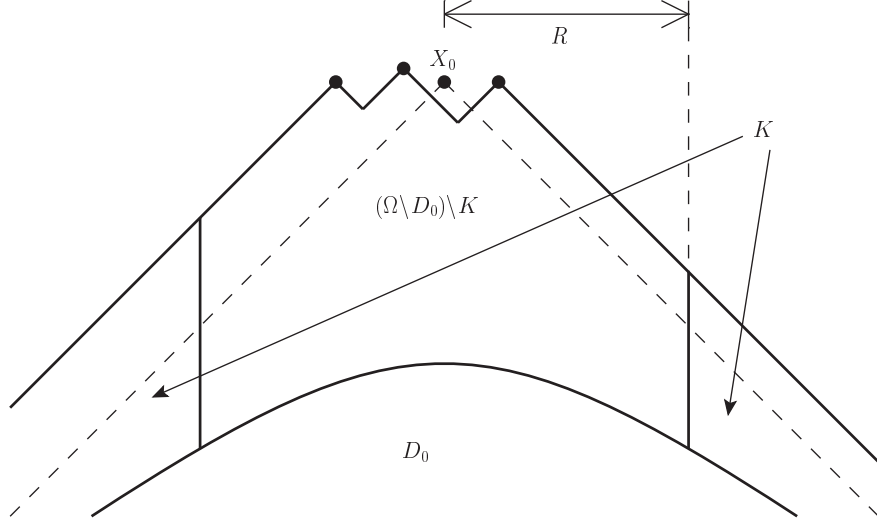


Figure 6: This figure is a schematic of how we divide the integration region Ω . There are D_0 , K and $(\Omega \setminus D_0) \setminus K$. The dots except for X_0 represent the external points. The dashed lines represent schematically the “past light cone of X_0 .”

the region K . Therefore, if $b'/(|-\eta_0|)$ is taken sufficiently large compared with Z_0 , the above range of η covers the whole region of K . Thus, the volume $\int_K |dV_Y|$ is bounded by

$$\int_K |dV_Y| < c_1 \int_{||\mathbf{x}_0 - \mathbf{y}|| > R} d^{D-1}y \int_{\eta_{i(\mathbf{y}),-} - b'}^{\eta_{i(\mathbf{y}),-} + b} \frac{|d\eta|}{|\eta|^D} < c_2 \int_{||\mathbf{x}_0 - \mathbf{y}|| > R} \frac{d^{D-1}y}{||\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}||^D} < +\infty, \quad (4.29)$$

where c_1 and c_2 are some appropriately chosen constants of $O(1)$. In the second inequality we used $|\eta| > |\eta_{i(\mathbf{y}),-} + b| \approx ||\mathbf{x}_{i(\mathbf{y})} - \mathbf{y}||$. Therefore, the integral over K is proven to be finite.

(ii) Integral over D_0 :

We next proceed to the integral over D_0 . Using Eq. (4.21), one can easily bound the volume integral of our current concern from above as

$$\int_{D_0} |dV_Y| \left| \left(\frac{1 - Z_{1Y}}{2} \right) \right|^{\text{Re } u_1} \dots \left| \left(\frac{1 - Z_{NY}}{2} \right) \right|^{\text{Re } u_N} < c_3 \times 2^{-\lambda} \int_{D_0} |dV_Y| |Z_{0Y}|^{\text{Re } \lambda}. \quad (4.30)$$

where c_3 is a constant of $O(1)$, and we have used the relation $\sum^N u_i = \lambda$.

In order to show this integral is finite, we use Z_{0Y} as a time coordinate instead of η , which leads the integration measure to transform as

$$d\eta d^{D-1}y = \left(1 - \frac{\eta_0 Z_{0Y}}{\sqrt{||\mathbf{x}_0 - \mathbf{y}||^2 + \eta_0^2 (Z_{0Y}^2 - 1)}} \right) \eta_0 d(Z_{0Y}) d^{D-1}y. \quad (4.31)$$

We substitute this into the right hand side of (4.30). Approximating $Z_{0Y}^2 - 1 \approx Z_{0Y}^2$ and introducing $x := y/(-\eta_0 Z_{0Y})$, we find that the integral is finite as

$$\int_{D_0} |dV_Y| |Z_{0Y}|^{\text{Re } \lambda} < c_4 \int_{Z_0}^{+\infty} \frac{dZ}{Z^{1-\text{Re } \lambda}} \int_0^{+\infty} dx \frac{1}{x/\sqrt{1+x^2}} \left(\frac{x}{(1+\sqrt{1+x^2})} \right)^{D-1} < +\infty, \quad (4.32)$$

where again c_4 is a constant of $O(1)$.

(iii) Summary:

We have shown in this subsection that the integral (4.12) is indeed finite when the external points X_1, \dots, X_n and X_N lie on the real Lorentzian section and are mutually in spacelike separation, as long as the integration contours for u_1, \dots, u_n satisfy the additional conditions (4.4) and (4.24):

$$\begin{aligned} \operatorname{Re} \lambda &= \operatorname{Re} \sum_i^N u_i < 0, \\ \operatorname{Re} u_1 &> -\frac{D}{2}, \quad \dots, \quad \operatorname{Re} u_n > -\frac{D}{2}, \quad \operatorname{Re} u_N > -\frac{D}{2}. \end{aligned} \quad (4.33)$$

Then, the order of two integrals $\int_{\Omega} dV_Y$ and $\int_C \prod_{i=1}^n du_i / 2\pi i$ in Eq. (4.5) are exchangeable, which implies that the master integral \mathcal{M} is given by the repeated Mellin transform of \mathcal{A} . Furthermore, under these conditions $\mathcal{A}(\alpha_1, \dots, \alpha_n)$ is finite and thus from Eq. (4.7) the master integral $\mathcal{M}(u_1, \dots, u_n, u_N)$ is also finite. That is, the master integral $\mathcal{M}(u_1, \dots, u_n, u_N)$ is finite when the external points are in the real Lorentzian section and are mutually in spacelike separation, with the conditions (4.33) satisfied. The analytic expression for \mathcal{M} is given in the succeeding subsection, where the conditions on the external points are relaxed.

4.2 Calculation of the Generating Function

We now proceed to compute \mathcal{A} and hence \mathcal{M} , to show its equivalence to the analytic continuation of the Euclidean correlators. Again in this subsection we first assume that all the external points X_i lie on the real Lorentzian section and that they are mutually in spacelike separation. After that, we show that the time coordinates of the external points η_i in the obtained expression for \mathcal{M} can be analytically continued to any point on the in-in path.

The expression for \mathcal{A} given in Eq. (4.3) can be transformed into

$$\mathcal{A}(\alpha_1, \dots, \alpha_n) = \int_{\mathbf{R}^{D-1}} d^{D-1}y \int_{P_y} \frac{d\eta}{(-\eta)^D} 2^{-\lambda} \left(\sum_{i=1}^N \alpha_i - V \cdot Y \right)^{\lambda}, \quad (4.34)$$

where $V \cdot Y$ is an inner product of V and Y with respect to $(D+1)$ -dimensional Minkowski metric and

$$V = \sum_{i=1}^N \alpha_i X_i. \quad (4.35)$$

Notice that

$$V^0 + V^D = \sum_{i=1}^N \frac{\alpha_i}{-\eta_i} > 0, \quad \mathbf{V} = \sum_{i=1}^N \frac{\alpha_i}{-\eta_i} \mathbf{x}_i. \quad (4.36)$$

Setting

$$\tau := \frac{V^0 + V^D}{2} \eta, \quad \mathbf{R} := (V^0 + V^D) \mathbf{x} - \mathbf{V} = \sum_{i=1}^N \frac{\alpha_i}{-\eta_i} (\mathbf{y} - \mathbf{x}_i), \quad (4.37)$$

$V \cdot Y$ can be expressed as

$$V \cdot Y = -\tau + \frac{\mathbf{R}^2 - V \cdot V}{4} \frac{1}{\tau}, \quad (4.38)$$

where

$$V \cdot V := \eta_{ab} V^a V^b = \left(\sum_{i=1}^N \alpha_i \right)^2 + 2 \sum_{i < j}^N \alpha_i \alpha_j (Z_{ij} - 1) . \quad (4.39)$$

Thus, we obtain

$$\begin{aligned} \mathcal{A}(\alpha_1, \dots, \alpha_n) &= \int_{\mathbf{R}^{D-1}} d^{D-1}y \int_{P_{\mathbf{y}}} \frac{d\eta}{(-\eta)^D} 2^{-\lambda} \left(\sum_{i=1}^N \alpha_i - V \cdot Y \right)^\lambda \\ &= 2^{-\lambda} \int_{\mathbf{R}^{D-1}} d^{D-1}y \left(\frac{V^0 + V^D}{2} \right)^{D-1} \int_{P'_{\mathbf{y}}} d\tau (-\tau)^{-D} \left[\sum_{i=1}^N \alpha_i + \tau - \frac{\mathbf{R}^2 - V \cdot V}{4} \frac{1}{\tau} \right]^\lambda \\ &= 2\pi i \times 2^{-D+1-\lambda} \int_{\mathbf{R}^{D-1}} d^{D-1}R \int_{P'_{\mathbf{y}}} \frac{d\tau}{2\pi i} (-\tau)^{-D-\lambda} (-\tau + \tau_+)^{\lambda} (\tau - \tau_-)^{\lambda} . \end{aligned} \quad (4.40)$$

where $P'_{\mathbf{y}}$ is the scale-transformed path of $P_{\mathbf{y}}$ by a factor of $(V^0 + V^D)/2$, and

$$\tau_{\pm} := \frac{1}{2} \left\{ -F \pm \sqrt{\mathbf{R}^2 + J^2} \right\} , \quad F := \sum_{i=1}^N \alpha_i , \quad J^2 := 2 \sum_{i < j}^N \alpha_i \alpha_j (1 - Z_{ij}) . \quad (4.41)$$

As is mentioned at the beginning of this subsection, we have assumed that the external points are all mutually in spacelike separation, so that

$$J^2 > 0 , \quad \tau_{\pm} \in \mathbf{R} . \quad (4.42)$$

Changing the integration variable further to $\xi := \tau - \tau_-$, we obtain

$$\mathcal{A}(\alpha_1, \dots, \alpha_n) = -2\pi i \times 2^{-D+1-\lambda} \int_{\mathbf{R}^{D-1}} d^{D-1}R \int_C \frac{d\xi}{2\pi i} (A - \xi)^{-D-\lambda} (B - \xi)^{\lambda} \xi^{\lambda} , \quad (4.43)$$

where

$$A := -\tau_- = \frac{1}{2} \left\{ F + \sqrt{\mathbf{R}^2 + J^2} \right\} , \quad B := \tau_+ - \tau_- = \sqrt{\mathbf{R}^2 + J^2} , \quad (4.44)$$

are both positive, and C is the integration contour shown in Fig. 7, which corresponds to $P'_{\mathbf{y}}$ but the direction is reversed. (See Fig. 4 as a reference.) Notice that the path C does not have to respect the $i\epsilon$ prescription here, because this integral is convergent without relying on the $i\epsilon$ regulator.

In order to compute this integral, we use the formula

$$\begin{aligned} &\int_C \frac{d\xi}{2\pi i} (A - \xi)^{\alpha} (B - \xi)^{\beta} \xi^{\gamma} \\ &= \int_{\mu} \Gamma \left[\begin{matrix} -\alpha + \mu, -(\alpha + \beta + \gamma + 1) + \mu, -\mu, \alpha + \gamma + 1 - \mu \\ -\alpha, -\beta, -\gamma, \gamma + 1 \end{matrix} \right] A^{\mu} B^{\alpha + \beta + \gamma + 1 - \mu} , \end{aligned} \quad (4.45)$$

which is valid for $\text{Re}(\alpha + \beta + \gamma + 1) < 0$.

It is not difficult to verify (4.45). We denote this integral as $I(\alpha, \beta, \gamma)$. When $\text{Re} \gamma > -1$, the path C can be contracted to the forward and backward paths along the negative real axis. Noticing that only

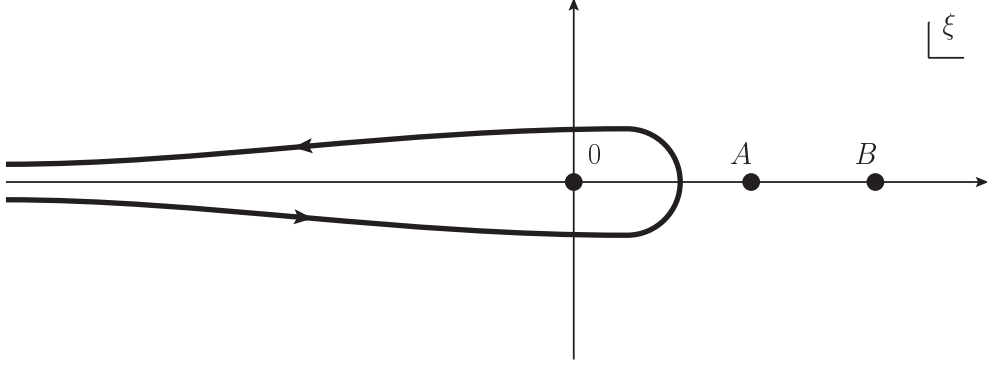


Figure 7: The path C in the integral (4.45).

the argument of ξ^γ changes between these two paths, one can transform $I(\alpha, \beta, \gamma)$ as

$$\begin{aligned} I(\alpha, \beta, \gamma) &= \frac{1}{2\pi i} \int_{+\infty}^0 (-dx) e^{-i\pi\gamma} x^\gamma (x+A)^\alpha (x+B)^\beta + \frac{1}{2\pi i} \int_0^{+\infty} (-dx) e^{i\pi\gamma} x^\gamma \dots \\ &= \frac{1}{\Gamma[-\gamma, \gamma+1]} \int_0^{+\infty} dx (x+A)^\alpha (x+B)^\beta x^\gamma. \end{aligned} \quad (4.46)$$

Next, we expand $(x+B)^\beta$, using Eq. (A.2), as

$$(x+B)^\beta = \int_\mu \Gamma \left[\begin{matrix} -\beta + \mu, -\mu \\ -\beta \end{matrix} \right] x^\mu B^{\beta-\mu}, \quad (\text{Re } \beta < \text{Re } \mu < 0). \quad (4.47)$$

Substituting this into Eq. (4.46), we carry out x integral first to obtain

$$I(\alpha, \beta, \gamma) = \int_\mu \Gamma \left[\begin{matrix} -\beta + \mu, -\mu, \gamma+1+\mu, -\alpha-\gamma-1-\mu \\ -\gamma, \gamma+1, -\beta, -\alpha \end{matrix} \right] A^{\alpha+\gamma+1+\mu} B^{\beta-\mu}. \quad (4.48)$$

Of course, the convergence of x integral imposes a condition $\text{Re}(\alpha + \gamma + 1 + \mu) < 0$. This is in fact satisfied because we can set $\text{Re } \mu$ arbitrarily close to $\text{Re } \beta$ as long as $\text{Re } \beta < \text{Re } \mu (< 0)$ is maintained. If we change the integration variable from $\mu \rightarrow \mu - \alpha - \gamma - 1$, we obtain the expression (4.45).

Finally, we remove the restriction $\text{Re } \gamma > -1$. In fact, the integrand is analytic for γ and the ξ integration is uniformly convergent for γ , as long as $\text{Re}(\alpha + \beta + \gamma + 1) < 0$. Therefore, the integral is analytic for γ , which enables us to remove the restriction $\text{Re } \gamma > -1$ by analytic continuation.

Substituting Eq. (4.45) into Eq. (4.43), we find

$$\begin{aligned} \mathcal{A}(\alpha_1, \dots, \alpha_n) &= -2\pi i \times 2^{-D-\lambda+1} \int_{\mathbf{R}^{D-1}} d^{D-1}R \int_\mu \Gamma \left[\begin{matrix} D+\lambda+\mu, D-\lambda-1+\mu, -\mu, -D+1-\mu \\ D+\lambda, -\lambda, -\lambda, \lambda+1 \end{matrix} \right] \\ &\quad \times 2^{-\mu} \left[F + \sqrt{\mathbf{R}^2 + J^2} \right]^\mu (\mathbf{R}^2 + J^2)^{\frac{-D+\lambda+1-\mu}{2}}. \end{aligned} \quad (4.49)$$

We next carry out \mathbf{R} integration, using the formula

$$\begin{aligned} &\int_{\mathbf{R}^{D-1}} d^{D-1}R [\mathbf{R}^2 + J^2]^{\nu/2} \left[F + \sqrt{\mathbf{R}^2 + J^2} \right]^\mu \\ &= \pi^{\frac{D-1}{2}} \int_\kappa \Gamma \left[\begin{matrix} -\mu + \kappa, -\kappa, -\frac{\kappa+\nu+D-1}{2} \\ -\mu, -\frac{\kappa+\nu}{2} \end{matrix} \right] F^{\mu-\kappa} J^{\nu+\kappa+D-1}, \end{aligned} \quad (4.50)$$

which is valid when $\text{Re}(\mu + \nu + D - 1) < 0$. The idea of the proof of the above formula is not so different

from that of the formula (4.45). One applies Eq. (A.2) to $\left[F + \sqrt{\mathbf{R}^2 + J^2}\right]^\mu$ to obtain

$$\left[F + \sqrt{\mathbf{R}^2 + J^2}\right]^\mu = \int_{\kappa} \Gamma \begin{bmatrix} -\mu + \kappa, -\kappa \\ -\mu \end{bmatrix} F^{\mu-\kappa} (\mathbf{R}^2 + J^2)^{\kappa/2}, \quad (\text{Re } \mu < \text{Re } \kappa < 0). \quad (4.51)$$

Substituting this into the left hand side of Eq. (4.50), we obtain

$$\begin{aligned} & \int_{\mathbf{R}^{D-1}} d^{D-1}R [\mathbf{R}^2 + J^2]^{\nu/2} \left[F + \sqrt{\mathbf{R}^2 + J^2}\right]^\mu \\ &= \Omega_{D-2} \int_{\kappa} \Gamma \begin{bmatrix} -\mu + \kappa, -\kappa \\ -\mu \end{bmatrix} F^{\mu-\kappa} J^{\nu+\kappa+D-1} \times \frac{1}{2} \int_0^{+\infty} d\Xi \Xi^{\frac{D-3}{2}} (1 + \Xi)^{\frac{\nu+\kappa}{2}}, \end{aligned} \quad (4.52)$$

where we have introduced a new integration variable $\Xi := (|\mathbf{R}|/J)^2$, and

$$\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}, \quad (4.53)$$

is the surface area of $D-2$ dimensional unit sphere. The Ξ integral is convergent if $\text{Re}(\nu + \kappa + D - 1) < 0$, which can be satisfied since we can choose $\text{Re } \kappa$ arbitrarily close to $\text{Re } \mu$ as long as $\text{Re } \mu < \text{Re } \kappa (< 0)$ is maintained. Integration over Ξ leads to (4.50).

Applying the formula (4.50) to the expression for the generating function (4.49), and replacing the integration variables κ and μ , respectively, with

$$w := \frac{\kappa - \mu + \lambda}{2} \quad \text{and} \quad \rho := \mu + D - 1, \quad (4.54)$$

we obtain

$$\begin{aligned} \mathcal{A}(\alpha_1, \dots, \alpha_n) &= (-i) 2^{2-\lambda} \pi^{\frac{D+1}{2}} \int_w \Gamma \begin{bmatrix} 2w - \lambda, -w \\ -w + \frac{D-1}{2}, D + \lambda, -\lambda, -\lambda, \lambda + 1 \end{bmatrix} \\ &\times \int_{\rho} 2^{-\rho} \Gamma[-\lambda + \rho, \lambda + 1 + \rho, -\rho, \lambda + D - 1 - 2w - \rho]. \end{aligned} \quad (4.55)$$

Finally, we perform the ρ integration in the above expression for \mathcal{A} , using the formula

$$\int_{\rho} 2^{-\rho} \Gamma[\lambda + 1 + \rho, -\lambda + \rho, -\rho, \lambda + a - 1 - \rho] = \frac{2^{\lambda+a-2}}{\sqrt{\pi}} \Gamma \left[-\lambda, \lambda + 1, \frac{a-1}{2}, \lambda + \frac{a}{2} \right], \quad (4.56)$$

which can be proven as follows. If we close the ρ -path on the left hand side of (4.56) to the right, we have

$$\begin{aligned} & \int_{\rho} 2^{-\rho} \Gamma[\lambda + 1 + \rho, -\lambda + \rho, -\rho, \lambda + a - 1 - \rho] \\ &= \Gamma[-\lambda, \lambda + 1, \lambda + a - 1] {}_2F_1 \left(-\lambda, \lambda + 1; 2 - \lambda - a; \frac{1}{2} \right) \\ &\quad + 2^{1-\lambda-a} \Gamma[2\lambda + a, a - 1, 1 - \lambda - a] {}_2F_1 \left(2\lambda + a, a - 1; \lambda + a; \frac{1}{2} \right). \end{aligned} \quad (4.57)$$

Now applying the following formulae, known respectively as Bailey's summation theorem and the Gauss'

second summation theorem [46],

$${}_2F_1\left(\alpha, 1-\alpha; \gamma; \frac{1}{2}\right) = 2^{1-\gamma} \sqrt{\pi} \Gamma\left[\frac{\gamma+\alpha}{2}, \frac{\gamma+(1-\alpha)}{2}\right], \quad (4.58)$$

$${}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; \frac{1}{2}\right) = \sqrt{\pi} \Gamma\left[\alpha+\frac{1}{2}, \beta+\frac{1}{2}\right], \quad (4.59)$$

we obtain after simple calculations (4.56). Substituting $a = D - 2w$ in (4.56), we find

$$\mathcal{A}(\alpha_1, \dots, \alpha_n) = (-i)(4\pi)^{D/2} \int_w \Gamma\left[\begin{matrix} 2w-\lambda, -w, \lambda+\frac{D}{2}-w \\ D+\lambda, -\lambda \end{matrix}\right] F^{\lambda-2w} \left(\frac{J}{2}\right)^{2w}. \quad (4.60)$$

Recalling the definition of F and J , (4.41), we expand $F^{\lambda-2w} J^{2w}$ to be integrals with respect to the power law indices of α_i 's using Eq. (A.1) [10]. Since the master integral \mathcal{M} is given by the Mellin transform of the generating function \mathcal{A} , we finally obtain

$$\begin{aligned} \mathcal{M}(\nu_1, \dots, \nu_N) &= (-i) \frac{(4\pi)^{D/2}}{\Gamma(D + \sum \nu_i) [\prod \Gamma(-\nu_i)]} \int_{(h_{ij})} \left[\prod_{i < j} \left(\frac{1-Z_{ij}}{2}\right)^{h_{ij}} \Gamma(-h_{ij}) \right] \\ &\quad \times \left[\prod \Gamma(H_i - \nu_i) \right] \Gamma\left(\frac{D}{2} + \sum \nu_i - \sum h_{ij}\right), \end{aligned} \quad (4.61)$$

where $\int_{(h_{ij})}$ represents $N(N-1)/2$ -fold integration $\prod_{1 \leq i < j \leq N} (\int dh_{ij}/2\pi)$, and

$$H_i := \sum_{k=1}^{i-1} h_{ki} + \sum_{k=i+1}^N h_{ik}. \quad (4.62)$$

In the above derivation of the equivalence between the expressions (4.1) and (4.61) we assumed that all external points are mutually in spacelike separation. However, we can easily extend the result (4.61) to the case of timelike separation. First, notice that the integrand of Eq. (4.1) is analytic for the time coordinates of the external points η_i and also that this Ω integral (4.1) continues to be well-defined and uniformly convergent even if η_i are analytically continued to the region of timelike separation. On the other hand, the (h_{ij}) integrals in (4.61) are convergent as long as $|\arg(1-Z_{ij})| < \pi$. This condition is satisfied when η_i are placed on the original path P . Later, we need to replace X_j to Y and then Y is placed on the path P_y . Even in this case it can be easily verified that the conditions $|\arg(1-Z_{ij})| < \pi$ are satisfied. As a result, by the uniqueness of the analytic continuation, the master integral (4.1) is identical to the expression (4.61) even if the separations between some pairs of external points are timelike. As a remark already mentioned at the end of the preceding subsection, the parameters $\nu_1, \dots, \nu_n, \nu_N$ must satisfy the conditions (4.33) in order for $\mathcal{M}(\nu_1, \dots, \nu_n, \nu_N)$ to be defined.

Furthermore, without violating the convergence conditions, we can continue the external points in (4.61) to the Euclidean region where $\arg(1-Z_{ij}) = 0$. Then, we find that the expression (4.61) is identical to the one obtained in the Euclidean Field Theory in Ref. [10], except for the factor of $-i$ due to convention.

5 Interacting QFT: Arbitrary Graphs

In the preceding section, we computed the master integral for a massive scalar field using the in-in formalism in the Lorentzian de Sitter space with the $i\epsilon$ prescription assuming the Euclidean vacuum at the level of non-interacting theory. We found that the resulting master integral is the analytic continuation of the one computed by the Euclidean path integral. Then, it might be expected that these two perturbative correlators are equivalent to all orders of perturbation. We will prove this equivalence along our

formulation in this section. Note that this equivalence is already shown to all orders, graph by graph in Ref. [11], in a strictly different way from the present paper.

5.1 Statement to be proven by induction

It is known that the Euclidean path integral gives us a certain analytic form corresponding to any graph $\mathcal{V}_N(X_1, \dots, X_N)$, which contributes to the N -pt correlator. The analytic expression for \mathcal{V}_N is found in Ref. [10] in the form

$$\mathcal{V}_N(X_1, \dots, X_N) = \int_{(h_{ij})} \left[\prod_{i < j}^N \left(\frac{1 - Z_{ij}}{2} \right)^{h_{ij}} \Gamma(-h_{ij}) \right] V_N(h_{ij}), \quad (5.1)$$

where $V_N(h_{ij})$ satisfies the following properties:

1. The fundamental strip for each variable h_{ij} of $V_N(h_{ij})$ contains the region

$$\text{Re } h_{ij} \in (\sigma - \mathcal{P}_{ij}(h'), 0], \quad (5.2)$$

where \mathcal{P}_{ij} is a linear combination of $\text{Re } h_{kl}$ excluding $\text{Re } h_{ij}$ with non-negative coefficients¹.

2. When h_{ij} is in the region (5.2), $V_N(h_{ij})$ falls off, for fixed h_{kl} except for h_{ij} , as rapidly as

$$V_N(\dots, h_{ij} = x + iy, \dots) \rightarrow e^{-\pi|y|/2} |y|^{x-1} \quad (|y| \gg 1). \quad (5.3)$$

In this section we shall show by induction that any correlators calculated in the in-in formalism have the same analytic form as the above obtained in the Euclidean path integral.

We start with some $(N+K)$ -pt correlator $\mathcal{V}_{N+K}(X_1, \dots, X_{N+K})$ which satisfies the properties 1 and 2 above. The succeeding steps are as follows:

- (a) Set K external points, X_{N+1}, \dots, X_{N+K} , in \mathcal{V}_{N+K} to Y .
- (b) Add $M - N$ propagators connected to Y and integrate over Ω with respect to Y , which gives a new M -point correlator with more loops.

Any graphs can be obtained by this construction, except for the ones containing “one-link” loops, which are to be renormalized. It has been already shown in Ref. [10] that the intermediate $(N+1)$ -pt function obtained in step (a) satisfies the properties 1 and 2. Therefore, what we have to consider is step (b). The resulting correlator, $\mathcal{V}_M(X_1, \dots, X_M)$, is given by

$$\mathcal{V}_M(X_1, \dots, X_M) = \int_{\Omega} dV_Y \mathcal{V}_{N+1}(X_1, \dots, X_N, Y) G(X_{N+1}, Y) \cdots G(X_M, Y). \quad (5.4)$$

Integration region Ω is specified in the same manner as in Sec. 3.1, but now with M external points, X_1, \dots, X_M . We show below that the M -pt correlator given in Eq. (5.4) has the form of Eq. (5.1).

5.2 Proof

We here set the external points in Eq. (5.4), X_1, \dots, X_M , to lie on the real Lorentzian section with mutually spacelike separation for technical reasons as in Sec. 4. Once we succeed in proving that $\mathcal{V}_M(X_1, \dots, X_M)$ in Eq. (5.4) satisfies the properties 1 and 2, it is obvious that the time coordinates η_I ($I = 1, \dots, M$) in $\mathcal{V}_M(X_1, \dots, X_M)$ can be analytically continued to the timelike separation or the Euclidean region for the same reason as we discussed for \mathcal{M} in the preceding section.

¹In Ref. [10], \mathcal{P}_{ij} is set to be “a polynomial function of all $\text{Re } h_{kl}$ except for $\text{Re } h_{ij}$ with non-negative coefficients,” which does not matter here.

Representing the respective factors in (5.4) in the Mellin-Barnes form, i.e., (2.9) for G 's and (5.1) for \mathcal{V}_{N+1} , we obtain

$$\begin{aligned} \mathcal{V}_M &= \int_{\Omega} dV_Y \int_{(h_{ij})} \int_{[\nu_i]} \left[\prod_{i < j}^N \left(\frac{1 - Z_{ij}}{2} \right)^{h_{ij}} \Gamma(-h_{ij}) \right] \left[\prod_{i=1}^N \left(\frac{1 - Z_{iY}}{2} \right)^{\nu_i} \Gamma(-\nu_i) \right] V_{N+1}(h_{ij}, \nu_i) \\ &\quad \times \int_{[\nu_{I'}]} \left[\prod_{I'=N+1}^M \left(\frac{1 - Z_{I'Y}}{2} \right)^{\nu_{I'}} \Gamma(-\nu_{I'}) \psi(\nu_{I'}) \right], \end{aligned} \quad (5.5)$$

where we have set the variables in the Barnes integral of $G(X_{I'}, Y)$'s ($I' = N+1, \dots, M$) to $\nu_{I'}$ and those for \mathcal{V}_{N+1} to h_{ij} ($1 \leq i < j \leq N+1$), and we replaced $h_{i, N+1}$ ($1 \leq i \leq N$) with ν_i . Here, we have denoted in short,

$$\int_{[\nu_i]} (\dots) := \int_{\nu_1} \dots \int_{\nu_N} (\dots), \quad (5.6)$$

and so forth. The integrals for h_{ij} and ν_i is a multiple integral and here we refer to the integration region for them as C . We rewrite \mathcal{V}_M above, using C , as

$$\begin{aligned} \mathcal{V}_M &= \int_{\Omega} dV_Y \int_C \prod_{i < j}^N \frac{dh_{ij}}{2\pi i} \prod_{i=1}^M \frac{d\nu_I}{2\pi i} \\ &\quad \left[\prod_{i < j}^N \left(\frac{1 - Z_{ij}}{2} \right)^{h_{ij}} \Gamma(-h_{ij}) \right] \left[\prod_{I=1}^M \left(\frac{1 - Z_{IY}}{2} \right)^{\nu_I} \Gamma(-\nu_I) \right] V_{N+1}(h_{ij}, \nu_i) \left[\prod_{I'=N+1}^M \psi(\nu_{I'}) \right]. \end{aligned} \quad (5.7)$$

Now the question is whether Ω integration and C integration are exchangeable. In order to examine it, we take the absolute value of the integrand and repeatedly integrate it to see whether the integral is finite or not. Here we consider the following repeated integral

$$\begin{aligned} &\int_C \prod \left| \frac{dh_{ij}}{2\pi i} \right| \prod \left| \frac{d\nu_I}{2\pi i} \right| \left| \prod \left(\frac{1 - Z_{ij}}{2} \right)^{h_{ij}} \Gamma(-h_{ij}) \right| |V_{N+1}(h_{ij}, \nu_i)| \left[\prod |\psi(\nu_{I'})| \right] \\ &\quad \times \left[\int_{\Omega} |dV_Y| \left| \prod \left(\frac{1 - Z_{IY}}{2} \right)^{\nu_I} \Gamma(-\nu_I) \right| \right], \end{aligned} \quad (5.8)$$

where we have dropped the indices for \prod and \sum . As a default, the ranges of various indices are understood as $1 \leq i \leq N$, $N+1 \leq I' \leq M$ and $1 \leq I \leq M$.

In order to evaluate Ω integration of the above expression, we apply the discussion in Sec. 4.1, but slightly modify it. In Sec. 4.1, the essential point is the bound for $|\arg(1 - Z_{iY}) - \arg(1 - Z_{NY})|$ because, in Sec. 4.1, the exponents, u_i , of $(1 - Z_{iY})$'s in the integrand are not independent since $\sum^N u_i = \lambda$.

In this subsection, however, they are mutually independent. Therefore, we should evaluate $|\arg(1 - Z_{IY})|$ itself. In fact, noting that the part dependent on the external points of the integrand of the Ω integration is expressed as

$$\prod |(1 - Z_{IY})|^{\text{Re } \nu_I} \exp \left[- \sum \arg(1 - Z_{IY}) \text{Im } \nu_I \right], \quad (5.9)$$

we have to bound $|\arg(1 - Z_{IY})|$ for our purpose.

Since the regions on the η -plane where $|\arg(1 - Z_{IY})| = \pi$ are half lines going from $-\infty$ to $\eta_{I,-}$ or from $\eta_{I,+}$ to $+\infty$ on the real axis, $|\arg(1 - Z_{IY})|$ is obviously less than π if $Y \in \Omega$. Furthermore, since the path $P_{\mathbf{y}}$ on the η -plane is, by definition, tilted by ϵ in the far past and deviates finitely from the region above, $|\arg(1 - Z_{IY})|$ is bounded by $\pi - \delta'$ with δ' being some finite positive constant. Therefore,

we can factor out the $\text{Im } \nu_I$ dependent part in Eq. (5.9) to obtain the bound

$$[\text{Eq. (5.9)}] < \prod |(1 - Z_{IY})|^{\text{Re } \nu_I} \exp \left[(\pi - \delta') \sum |\text{Im } \nu_I| \right] . \quad (5.10)$$

Now the convergence of the volume integral follows in the exactly same manner as before. Thus, we are led to discuss the following integral

$$\begin{aligned} & \int_C \prod \left| \frac{dh_{ij}}{2\pi i} \right| \prod \left| \frac{d\nu_I}{2\pi i} \right| \left| \prod \left(\frac{1 - Z_{ij}}{2} \right)^{h_{ij}} \Gamma(-h_{ij}) \right| \\ & \times |V_{N+1}(h_{ij}, \nu_i)| \left[\prod |\psi(\nu_{I'})| \right] \exp \left[(\pi - \delta') \sum |\text{Im } \nu_I| \right] |\Gamma[-\nu_1, \dots, -\nu_N]| . \end{aligned} \quad (5.11)$$

Recall that $\psi(\nu)$ behaves as

$$|\psi(x + iy)| \rightarrow e^{-3\pi|y|/2} |y|^{x-1} , \quad (|y| \gg 1) , \quad (5.12)$$

and $|\Gamma(x + iy)| \approx (2\pi)^{1/2} e^{-\pi|y|/2} |y|^{x-1/2} (|y| \rightarrow +\infty)$. Furthermore, V_{N+1} behaves as

$$|V_{N+1}(h_{12} = x + iy, \dots)| \rightarrow e^{-\pi|y|/2} |y|^{x-1} , \quad (|y| \gg 1) , \quad (5.13)$$

from the property 2 of the assumption of induction, and the same is true for the other arguments, too. Therefore, the integral (5.11) is convergent, and hence the order of the integration over Ω and C in (5.7) are exchangeable:

$$\begin{aligned} \mathcal{V}_M &= \int_{(h_{ij})} \int_{[\nu_I]} \left[\prod \left(\frac{1 - Z_{ij}}{2} \right)^{h_{ij}} \Gamma(-h_{ij}) \right] V_{N+1}(h_{ij}, \nu_i) \left[\prod \psi(\nu_{I'}) \right] \\ &\times \int_{\Omega} dV_Y \left[\prod_{I=1}^M \left(\frac{1 - Z_{IY}}{2} \right)^{\nu_I} \Gamma(-\nu_I) \right] . \end{aligned} \quad (5.14)$$

Now substituting the Mellin-Barnes form for the master integral, (4.61), into Eq. (5.14), we arrive at the same Mellin-Barnes representation for \mathcal{V}_M as that obtained in Sec. 4.2 of Ref. [10], where \mathcal{V}_M is shown to be represented in the form of Eq. (5.1) with V_M satisfying the properties 1 and 2. This completes the proof of the equivalence between the two types of correlators.

6 Summary

In this work, we considered massive interacting scalar field theory and demonstrated perturbative calculation for the correlators using in-in formalism in the flat chart of de Sitter space with the $i\epsilon$ prescription. We found that the master integral defined in Eq. (3.7) has completely the same Mellin-Barnes representation as that obtained in Ref. [10] based on the Euclidean field theory. We then derived the analytic Mellin-Barnes formulae for the correlators of quantum field on the flat chart. The resulting correlators are shown to be completely the same as the analytic continuations of the ones considered in the Euclidean field theory. Thus we find that the $i\epsilon$ prescription in de Sitter space gives the Euclidean vacuum.

Although the relation between these two vacua has been clarified in Ref. [11], in order to extend this to massless field theory, we gave an alternative proof of their equivalence by direct calculation. In particular, graviton in de Sitter space has been a topic of much discussion. (See, e.g. [13, 21–23].) It is also worth considering derivatively interacting massless scalar field as a model of graviton.

The proof in Ref. [11] of the equivalence between the two vacua relies on the decay of the propagator at a large separation. But the propagators in massless theory do not fall off in general. This could be an obstacle in extending the discussion to interacting massless field theory. Though we considered only massive theory in this work, we believe that our proof has potential to be extended to wider range of

theories which include derivatively interacting massless field theory, since our proof of the correspondence of the correlators is based on direct calculation without relying on this property.

Acknowledgments

We would like to thank H. Kitamoto and V. Onemli for valuable comments. YK also thanks to R. Saito, K. Sugimura and K. Nakata for useful and interesting discussions. YK is supported by the Grant-in-Aid for JSPS Fellows No. 24-4198. TT is supported by the Grand-in-Aid for Scientific Research Nos. 21111006, 21244033, 24103001 and 24103006. This work was also supported by the Grant-in-Aid for the Global COE programs, “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

A Formula

Let A_1, \dots, A_{n+1} be complex numbers satisfying $|\arg A_i - \arg A_j| < \pi$ ($\forall i, j$). Then, the following formula is true as a repeated integral and also as a multiple integral since the integral is easily shown to be independent of the order of the integration:

$$\begin{aligned} & (A_1 + A_2 + \dots + A_{n+1})^\lambda \\ &= \frac{1}{\Gamma(-\lambda)} \int_{u_1} \dots \int_{u_n} \Gamma\left[-\lambda + \sum u_i, -u_1, \dots, u_n\right] (A_1)^{u_1} \dots (A_n)^{u_n} (A_{n+1})^{\lambda - \sum u_i} . \end{aligned} \quad (\text{A.1})$$

Proof of (A.1):

The basic formula is the following:

$$(a + b)^\lambda = \frac{1}{\Gamma(-\lambda)} \int_{\mu} \Gamma[-\lambda + \mu, -\mu] a^\mu b^{\lambda - \mu} , \quad (|\arg a - \arg b| < \pi) . \quad (\text{A.2})$$

One applies this formula (A.2) with $a = A_n$, $b = A_1 + \dots + A_{n-1} + A_{n+1}$, and then again apply (A.2) to $(A_1 + \dots + A_{n-1} + A_{n+1})^{\lambda - \mu}$ in the result of the previous step with $a = A_{n-1}$, $b = A_1 + \dots + A_{n-2} + A_{n+1}$. Repeating the same operation, one formally reaches (A.1). The point is that the conditions

$$\begin{aligned} & |\arg A_1 - \arg A_{n+1}| < \pi , \\ & |\arg A_2 - \arg(A_1 + A_{n+1})| < \pi , \\ & \dots \\ & |\arg A_n - \arg(A_1 + \dots + A_{n-1} + A_{n+1})| < \pi , \end{aligned} \quad (\text{A.3})$$

are required to perform the above transformation. To allow the exchange of the order of the repeated integration without changing the result, we impose stronger conditions

$$\begin{aligned} & |\arg A_{P(1)} - \arg A_{n+1}| < \pi , \\ & |\arg A_{P(2)} - \arg(A_{P(1)} + A_{n+1})| < \pi , \\ & \dots \\ & |\arg A_{P(n)} - \arg(A_{P(1)} + \dots + A_{P(n-1)} + A_{n+1})| < \pi , \end{aligned} \quad (\text{A.4})$$

for any permutation P . It is easily verified that, if we choose A_i 's to satisfy

$$|\arg A_i - \arg A_j| = |\arg(A_i/A_j)| < \pi/2 , \quad (\text{A.5})$$

for all pairs of i and j , the conditions (A.4) are all satisfied because in general $|\arg(\sum_I r_I)| < \pi/2$ if

$|\arg(r_I)| < \pi/2$ for all r_I where r_I 's are understood as A_i/A_j 's. However, this restriction can be easily relaxed by analytic continuation with respect to A_i as long as the conditions

$$|\arg A_i - \arg A_j| < \pi , \quad (\text{A.6})$$

are satisfied for all pairs since the right hand side of Eq. (A.1) continues to converge under these conditions. This completes the proof of (A.1).

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